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PROGRESS REPORT.

1. The following gentlemen have been elected members of the Society:

1. Mr. P. N. Natesh Aiyar, L.C.E., Executive Engineer, Ratnagiri;
2. Dr. H. N. Allen, B.Sc., PH.D., Principal and Professor of Physics and Electrical Engineering, College of Science, Poona.
3. Mr. Karpur Shrinivasa Rao, B.Sc., L.C.E., (Formerly Executive Engineer, Sindh) Superintending Engineer to Mysore Gevernment, Bangalore.
4. Mr. G. Chakrapani Rao, B.A., L.T., Mathematics Assistant Basava High School, Punganur Chittore District, (at concessional rate).

2. The Committee feel great pleasure in announcing to the General Body that the following two books have been presented to the Library by His Excellency the Governor of Bombay.

1. *The Modern Geometry of the Triangle* by Gallatly.
2. *Graphics* applied to Arithmetic, mensuration and statics by G. C. Turner (Macmillan & Co).

3. Messrs. G. Bell and Sons have presented to the Library—*A New Geometry* by Baker and Bourne.

POONA,
1st August 1911.

D. D. KAPADIA,
Hon. Joint Secretary.

On the locus of points at which opposite sides of a quadrilateral subtend equal or supplementary angles.

By M. T. Naraniengar, M.A.

Taking the harmonic triangle of a given quadrilateral as the triangle of reference the trilinear coordinates of the vertices of the quadrilateral are written $\{\pm d, \pm e, \pm f\}$.

1. *To find the trilinear equation of the locus.*

The tangent of the angle between two straight lines represented by

$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0,$$

is written

$$\pm \frac{\begin{vmatrix} l & m & n \\ l' & m' & n' \\ \sin A & \sin B & \sin C \end{vmatrix}}{ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (l'm' + l'm)\cos C} \qquad (1)$$

Let P, Q, R, S be the vertices of the quadrilateral corresponding to the coordinates.

$$(d, e, f),\ (-d, e, f),\ (d, -e, f),\ (d, e, -f).$$

Then the equation to the line joining P to any point X $(\alpha'\beta'\gamma')$ is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ d & e & f \end{vmatrix} = 0;$$

similarly, for the lines QX, RX, SX.

The tangent of the angle between PX and QX is given by (1), where l, m, n; l', m', n' denote respectively

$$\left\Vert \begin{matrix} \alpha' & \beta' & \gamma' \\ d & e & f \end{matrix} \right\Vert \qquad \left\Vert \begin{matrix} \alpha' & \beta' & \gamma' \\ -d & e & f \end{matrix} \right\Vert$$

For the tangent of the angle between RX, SX a similar result holds with suitable changes. Call this (2).

Hence, the locus required is

$$(1) \pm (2) = 0,$$

where the accents have to be dropped from $\alpha'\beta'\gamma'$. This denotes *apparently* two quartic curves.

2. *Simplification of the equation.*

Write x, y, z for $\alpha/d, \beta/e, \gamma/f$ respectively; then (1) and (2) reduce to

$$\pm\ 2(y \pm z)\ \{ dx \sin A + ey \sin B + fz \sin C \}\ /D$$

where D stands for $dx^2\left(\frac{e}{f}+\frac{f}{e}\right)+ey^2\left(\frac{f}{d}-\frac{d}{f}\right)+fz^2\left(\frac{e}{d}-\frac{d}{e}\right)\pm\frac{2efyz}{d}$ $-2d\cos A\,(yz\pm x^2)+2x(y\pm z)(f\cos C\pm e\cos B)$

The locus (1)±(2)=O, is thus two cubics and the straight line $dx\sin A+ey\sin B+fz\sin C=0$, which is the line at infinity.

Further simplyfying, we get the cubics in the forms

$$y^3\left(\frac{1}{d^2}-\frac{1}{f^2}\right)+y\left\{x^2\left(\frac{1}{f^2}+\frac{1}{e^2}\right)-z^2\left(\frac{1}{d^2}+\frac{1}{e^2}\right)\right\}+2\cos A\,\frac{z(x^2-y^2)}{ef}+\frac{2\cos C\,x(y^2-z^2)}{de}=0,$$

$$z^3\left(\frac{1}{e^2}-\frac{1}{d^2}\right)+z\left\{y^2\left(\frac{1}{d^2}+\frac{1}{f^2}\right)-x^2\left(\frac{1}{f^2}+\frac{1}{e^2}\right)\right\}+\frac{2\cos B\,x(y^2-z^2)}{df}+\frac{2\cos A\,y(z^2-x^2)}{ef}=0.$$

Substituting for x, y, z, and regrouping, the *first* equation is written

$$\frac{\beta}{d^2e^3}(\alpha^2+\beta^2+2\alpha\beta\cos C)-\frac{\beta}{f^2e^3}(\beta^2+\gamma^2+2\beta\gamma\cos A)+$$
$$+\frac{1}{d^2ef^2}\left\{(\beta(\alpha^2-\gamma^2)+2\alpha\gamma(\alpha\cos A-\gamma\cos C)\right\}=0$$

Write U for $\beta^2+\gamma^2+2\beta\gamma\cos A$, &c., and therefore $(U-\beta^2-\gamma^2)/2\beta\gamma$ for cosA, &c,. Then the equation becomes

$$\frac{\beta W}{d^2e^3}-\frac{\beta U}{d^2f^3}+\frac{1}{\beta}\frac{(\alpha^2U-\gamma^2W)}{d^2ef^2}=0$$

i.e. $U(e^2\alpha^2-d^2\beta^2)+W(f^2\beta^2-e^2\gamma^2)=0$

i.e. $d^2U(\alpha^2/d^2-\beta^2/e^2)+f^2W(\beta^2/e^2-\gamma^2/f^2)=0$

or $$\frac{d^2U}{y^2-z^2}+\frac{f^2W}{x^2-y^2}=0.$$

Similarly, the *second* equation is written

$$\frac{d^2U}{y^2-z^2}+\frac{e^2V}{z^2-x^2}=0.$$

Thus the locus corresponding to *every pair* of opposite sides of the quadrilateral PQRS is some two of the following three curves:

$$\frac{e^2V}{z^2-x^2}+\frac{f^2W}{x^2-y^2}=0,$$
$$\frac{f^2W}{x^2-y^2}+\frac{d^2U}{y^2-z^2}=0,$$
$$\frac{d^2U}{y^2-z^2}+\frac{e^2V}{z^2-x^2}=0.$$

It must be noted, that each of the above is *really* a cubic, the *linear* factors x, y, z respectively which appear on simplification, being rejected.

3. **We shall consider, as typical of the three, the curve**

$$e^2V/(z^2-x^2)+f^2W/(x^2-y^2)=0,$$

and discuss some of its properties.

4. *The curve is a circular cubic, circumscribing the given quadrilateral. and passing through B,C and the foot of the perpendicular from A on BC.*

Since V, W denote lines joining the points B, C with the circular points at infinity, it is clear that the equation of the curve is satisfied by the coordinates of the circular points at infinity. Hence, the cubic is a circular cubic, the factor $\alpha=0$ being rejected.

That the curve passes through the vertices of the quadrilateral PQRS is obvious, since at these vertices $x^2=y^2=z^2=1$. Also, the coordinates of B,C and the foot of the perpendicular from A, are easily seen to saticfy the equation of the curve.

5. *Interpretation of the equation.*

Write σ for $(a\alpha+b\beta+c\gamma)/2\Delta$, then the distances of the point P $(\alpha\beta\gamma)$ from the vertices of the triangle of reference are

$$(\beta^2+\gamma^2+2\,\beta\gamma\cos A)^{\frac{1}{2}}/\sigma, \text{ \&c.}$$

i.e. $$U^{\frac{1}{2}}/\sigma\,,\ V^{\frac{1}{2}}/\sigma\,,\ W^{\frac{1}{2}}/\sigma.$$

In fact U, V, W, denote *point circles* at the vertices of the triangle of reference and may be regarded as the equations of these vertices. Hence $\quad V : W = BX^2 : CX^2$.

Also, (x^2-y^2) varies as the product of the perpendiculars from X on the lines $x\pm y=0$, *viz.*, QS, PR; similarly for (z^2-x^2).

Thus the equation of the cubic, when interpreted, leads to the following theorem:

The ratio of the distances of any point of the cubic from B,C varies as the ratio of the Geometric Means of the perpendiculars from the point on the sides of the quadrilateral which meet at B,C respectively.

Again, if we write the equation in the form

$$x^2(e^2V-f^2W)-e^2y^2V+f^2z^2W=0,$$

we obtain

$$(e^2-f^2)x^2t_1^2-e^2y^2t_2^2+f^2z^2t_3^2=0$$

where t_1 denotes the tangent from X $(\alpha\beta\gamma)$ to the circle e^2V-f^2W, and t_2,t_3 denote BX, CX, *i.e.*, tangents to the point circles V, W.

In other words, the equation may be transformed into

$$lx^2t_1^2+my^2t_2^2+nz^2t_3^2=0$$

where $l+m+n=0$, and t_1, t_2, t_3 are tangents to the coaxal circles e^2V-f^2W, V,W respectively.

6. *The asymptote of the curve.*

This is readily found by writing the equation in the form

$$v_1S+I.v_2=0$$

where S is a circle, I the line at infinity and v_n a ternary quantic in α, β, γ; whence v_1 determines the *direction* of the asymptote. We shall, however, adopt a different plan, as follows:

The line at infinity cuts the curve in the two *imaginary* circular points and a third *real* point. The direction of the real point therefore gives the direction of the asymptote.

Now, putting $a\alpha = -(b\beta + c\gamma)$ in the equation of the cubic, and remembering that the circular points satisfy $\beta^2 + \gamma^2 + 2\beta\gamma \cos A = o$, we have

$$b\beta(a^2d^2 + c^2f^2 - b^2e^2) = c\gamma(a^2d^2 + b^2e^2 - c^2f^2).$$

for the *direction* required. The point at infinity corresponding to this direction being taken, the asymptote itself is obtained as the tangent to the cubic at the point at infinity.*

A special case.

7. Let the points PQRS in § 1 coincide with the *in* and *ex*centres of the triangle of reference; then $d^2 = e^2 = f^2 = 1$ and x^2, y^2, z^2 may be replaced by α^2, β^2, γ^2. Thus the equation of the cubic is reduced to

$$\frac{V}{\gamma^2 - \alpha^2} + \frac{W}{\alpha^2 - \beta^2} = 0.$$

This may also be written

$$\alpha\,(\beta^2 - \gamma^2) = \beta \cos C\,(\gamma^2 - \alpha^2) + \gamma \cos B\,(\alpha^2 - \beta^2).$$

8. *Some properties of the special cubic.*

The following properties are easily proved:

(i) The cubic circumscribes the triangle of reference.

(ii) It is its own isogonal transformation; that, is, if X is a point on it, the isogonal conjugate of X is also a point on it.

(ii) The tangents at A, B, C concur at the point of the circumcircle diametrically opposite to A.

(iv) The tangents at I, I_1, I_2, I_3 are parallel, being parallel to the asymptote of the curve.

(v) The asymptote is perpendicular to BC and passes through the point of the circumcircle diametrically opposite to A.

(vi) The curve passes through the points in which the lines drawn through A, B, C parallel to the asymptote, meet the opposite sides.

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* The asymptote may also be deduced by forming the cartesian equation of the curve referred to the opposite sides of the quad. as axes, which also furnishes an alternative method of investigating the properties of the cubic.

9. *The centres and circles of inversion of the cubic*:

By the well-known properties of the centres of inversion, *viz.*, that the tangents at the centres are parallel to the asymptote, it is clear that I, I_1, I_2, I_3 are the *four* centres of inversion[1].

The four corresponding circles of inversion are easily obtained. For, if through I a centre of inversion a chord IXX′ of the cubic is drawn the rectangle IX. IX′ = square of the radius of the circle of inversion $=\rho^2$ (say)[2]. Thus, the radii of the circles of inversion are given by

$$\rho^2 = IA.\ II_1 = -4\ Rr.$$
$$\rho_1^2 = I_1A.\ I_1I = +4\ Rr_1$$
$$\rho_2^2 = I_2B.\ I_2I = +4\ Rr_2$$
$$\rho_3^2 = I_3C.\ I_3I = +4\ Rr_3.$$

10. *The focal parabolas*:

A focal parabola of a circular cubic may be defined as follows—

If through a centre of inversion I, a chord IXX′ be drawn to cut the cubic in X, X′ the envelope of the straight line bisecting XX′ at right angles is a focal parabola of the cubic.[3]

Hence, if I be the centre of inversion the corresponding focal parabola is touched by the lines bisecting perpendicularly the segments AI_1, BI_2, CI_3; that is, the parabola is escribed to the medial triangle, and therefore its focus lies on the N.P. circle. Again, the tangent at I is parallel to the asymptote and perpendicular to BC; so that, $I\infty$ is a particular chord of the cubic and the corresponding tangent to the focal parabola is the tangent at infinity. In other words, the directrix of the parabola is parallel to the asymptote and therefore perpendicular to BC.

Hence, we have the property: *the directrix of the focal parabola corresponding to I, is the line through the circumcentre of $I_1I_2I_3$ perpendicular to BC.*

Now, it is readily seen that the pedal line of A with respect to the medial triangle of $I_1I_2I_3$ is perpendicular to BC and bisects AI_1. That is to say, the join of I, to the circumcentre of $I_1I_2I_3$ is perpendicular to BC. We are thus lead to the interesting results:

(i) *The focus of the (I) focal parabala is A.*

(ii) *The corresponding directrix is the line through I_1, perpendicular to BC; that is, the tangent at I_1 to the cubic.*

Similarly, the other three focal parabolas (corresponding to I_1, I_2, I_3) have the same focus A, and directrices perpendicular to BC through I, I_3, I_2 respectively.

[1] *Basset*: Cubic and Quartic Curves, § 235.

[2] *Basset*: Cubic and Quartic Curves, § 200.

[3] *Basset* § 200.

11. *The foci of the cubic* :

These are the points of intersection of the focal conic and the circle of inversion.[1] A geometrical construction for the foci may be given as follows :—

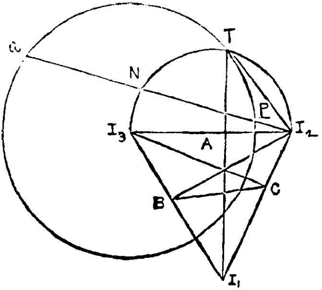


Fig. 1

Let the circle of inversion (I_3) be described, and tangent I_2 T drawn to it. Draw T N parallel to BC to cut the circle on I_2 I_3 as a diameter in N. Then I_2 N cuts the circle (I_3) in a pair of foci of the cubic. On joining I_1 to these to cut the circle again the remaining foci are found.

Proof: Suppose I_2 is a point on the directrix of a parabola whose focus is A. Draw any chord I_2 PQ, and let T be its pole. Then I_2 T touches the circle TPQ, as is easily proved. Thus the centre of the circle TPQ is I_3 on I_2 A produced, such that $I_2 A . I_2 I_3 = I_2 P . I_2 Q = I_2 T^2$

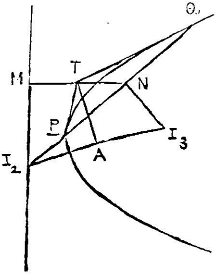


Fig. 2

[1] *Basset* § 211.

Now, if P, Q be a pair of foci of the cubic, they must be two of the intersections of the circle of inversion (I_3) and the corresponding focal parabola: they must also be collinear with one of the remaining centres of inversion I, I_1, I_2 (say I_2)[1]. Also, I_2 is a point on the directrix of the parabola by § 10.

Thus the chord PQ must be such that TN bisecting it is a diameter of the parabola; that is, is parallel to BC. And I_3 N is obviously prependicular to PQ.

Hence, the construction given above (Fig 1) for the foci P,Q of the cubic; the remaining foci are determined as suggested, since I I_1 I_2 is the *harmonic triangle* of the quadrilateral whose vertices are the four foci.

12. A tracing of the cubic is shown in Fig 3·

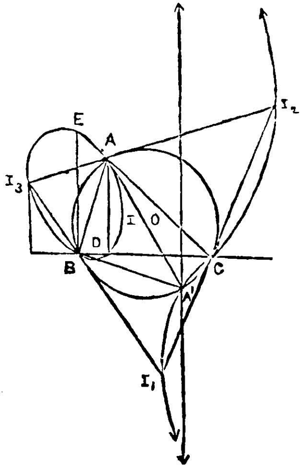


Fig. 3.

Another special case·

13. Suppose, the fundamental quadrilateral PQRS in § 1 is a parallelogram. Then, two of the vertices of the harmonic triangle will be at infinity; and we may employ Cartesian coordinates as follows:—

[1] *Basset*: § 2.1

Take A the intersection of the diagonals for origin, the axes being parallel to the sides of the parallelogram. The condition that PQ, SR should subtend equal or supplementary angles at a point (xy) is written in the form

$$\frac{2b \sin \omega\,(x-a)}{(x-a)^2+2(x-a)y \cos \omega+y^2-b^2} \pm \frac{2\,b \sin \omega\,(x+a)}{(x+a)^2+2(x+a)y \cos \omega+y^2-b^2}=0,$$

which reduces to

$$x(x^2+y^2-a^2-b^2)+2y(x^2-a^2) \cos \omega=0, \qquad (1)$$

or

$$x^2-y^2=a^2-b^2. \qquad (2)$$

Similarly, the locus corresponding to the remaining two sides of the parallelogram is

$$y(x^2+{}^2y-a^2-b^2)+2x(y^2-b^2) \cos \omega=0, \qquad (3)$$

or

$$x^2-y^2=a^2-b^2.$$

Hence, the locus generally is one of the three curves (1),(2),(3) above; the second is a conic, and the other two are circular cubics of the same kind, except when $\omega=90°$. It will be sufficient, therefore, if we discuss here the first curve in detail.

14. The curve is evidently symmetrical in opposite quadrants and its asymptote is the y–axis. It passes through the origin, which is a point of inflexion. The tangents at P, Q meet at T on the circle APQ, such that AT is isogonally conjugate to the x–axis with respect to PAQ; for, considering P as a particular point of the locus, the angle subtended by SQ at P is equal to that between PR and the tangent at P, etc.,

15. *The focal parabola.*

Transforming to rectangular coordinates, the y–axis remaining unchanged, the equation becomes

$$x(x^2+y^2)-x(b^2-a^2 \cos 2\omega)-a^2y \sin 2\omega=0 \qquad (4)$$

whence we can deduce that the *origin* is the *focus* of the *focal parabola.*

For, take the focus of a parabola of latus rectum $4p$ as origin, and let the centre and radius of inversion be (f,g), δ respectively. Then the equation to the corresponding circular cubic will be

$$(x+2p-f)\{(x-f)^2+(y-g)^2\}+2f(x-f)^2+2g(x-f)(y-g)$$
$$+\delta^2(x-f)=0;$$

when $f=2p$, and $\delta^2=2(4p^2+g^2)$, this reduces to

$$x(x^2+y^2)-x(4p^2-g^2)-4pgy=0$$

which is of form (4). Comparing the two, we have

$$\left.\begin{aligned} 4p^2-g^2&=b^2-a^2 \cos 2\omega,\\ 4pg&=a^2 \sin 2\omega. \end{aligned}\right\} \qquad (5)$$

Hence, we infer that A is the focus of the focal parabola; and that its directrix is parallel to the y—axis, or the asymptote.

16. *The centres of inversion.*

The centre of inversion being (f,g), where $f=2p$, we have to solve equations (5) to get the two centres O, O′. Analytically these are readily seen to be the points the tangents at which are parallel to the asymptote.

Again, the bisectors of the angles between the diagonals are given by

$$\frac{x^2-y^2}{b^2-a^2\cos 2\omega}=\frac{2xy}{a^2\sin 2\omega}, \qquad (6)$$

Therefore the points O, O′ determined by (5) lie on (6).

Further $(4p^2+g^2)^2=(b^2-a^2\cos 2\omega)^2+(a^2\sin 2\omega)^2$

$$=a^4+b^4-2a^2b^2\cos 2\omega$$
$$=(a^2+b^2+2ab\cos\omega)(a^2+b^2-2ab\cos\omega)$$
$$=AP^2.\ AQ^2.$$

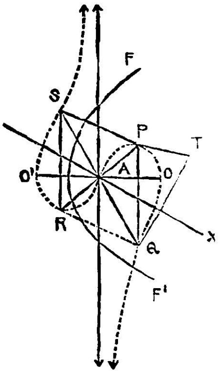
i.e., $AO^2=AP.\ AQ$

Hence, the centre of inversion O is such that the triangles PAO OAQ are *directly* similar.

Also, the radius δ of the circle of inversion is $\sqrt{2(4p^2+g^2)}=AO\sqrt{2}$

From the known property of the circle of inversion, we see that the tangent at O′ is the directrix of the focal parabola corresponding to the centre O of inversion.

17. The cubic in question is shown in Fig. 4, where FF′ indicates the focal parabola.



Some Metrical Relations connected with a pair of Isogonal Conjugates.

BY V. RAMASWAMI AIYAR, M.A.

1. Let ABC be a given triangle, O its circumcentre and P,P′ a pair of isogonal conjugates. We shall use the following notation:

R for the circumradius of ABC;

α,β,γ the semiaxis major, the semi-axis minor and the semi latus rectum of the inscribed conic whose foci are P,P′;

p,q,r the distances of P from the sides of ABC;

l,m,n the distances of P from the vertices;

$\lambda = R^2 - OP^2$;

$\rho = (R^2 - OP^2)/2R$; and

p',q',r', l',m',n', λ', ρ' for like quantities as regards P′.

We have thus introduced *twenty* quantities and between them the following *nine* relations exist, which are algebraically independent of one another:

$$pp' = qq' = rr' = \beta^2; \qquad \text{(A)}$$

$$\frac{l}{l'} = \frac{q}{r'},\ \frac{m}{m'} = \frac{r}{p'},\ \frac{n}{n'} = \frac{p}{q'}; \qquad \text{(B)}$$

$$\beta^2 = \alpha\gamma; \qquad \text{(C)}$$

$$\lambda = 2R\rho,\ \lambda' = 2R\rho' \qquad \text{(D)}$$

We shall investigate some others.

2. Let AA′,BB′,CC′ be chords of the circumcircle of ABC drawn through P; and let A_1, B_1, C_1 be the feet of the perpendiculars from P on the sides of ABC. The sides of triangle $A_1B_1C_1$ are $l \sin A$, $m \sin B$ $n \sin C$. The sides of the triangle A′B′C′ are readily seen to be $\frac{\lambda a}{mn}$, $\frac{\lambda b}{nl}$, $\frac{\lambda c}{lm}$. And it follows that the triangle $A_1B_1C_1$ is similar to A′B′C′ and that the ratio of similitude is $lmn/2R\lambda$.

3. *To show that* $lmn = 2\alpha\lambda$, *and* $l'm'n' = 2\alpha'\lambda'$. The feet of the perpendiculars from P,P′ on the sides, lie on the auxiliary circle of the inscribed conic whose foci are P,P′. The radius of this circle is therefore α; that is, α is the circumradius of the triangle $A_1B_1C_1$. And R is the circumradius of A′B′C′. Hence, by similitude, we have

$$\frac{\alpha}{R} = \frac{lmn}{2R\lambda}$$

or,

$$lmn = 2\alpha\lambda.$$

We have similarly $l'm'n' = 2\alpha\lambda'$.

Cor. Using the values of λ,λ' from (D) we get $lmn = 4R\alpha\rho$, $l'm'n' = 4R\alpha\rho'$.

4. **Let Δ_1 be the area of the pedal triangle $A_1B_1C_1$. To show that**

$$\Delta_1 = \tfrac{1}{2}(R^2 - OP^2) \sin A \sin B \sin C.$$

We have
$$\Delta_1 = \frac{\text{product of the sides of } A_1B_1C_1}{4 \text{ times circumradius}}$$
$$= \frac{lmn \sin A \sin B \sin C}{4a}$$
$$= \frac{2a\lambda \sin A \sin B \sin C}{4a}$$
$$= \tfrac{1}{2}(R^2 - OP^2) \sin A \sin B \sin C.$$

Cor. If Δ be the area of ABC, we have $\Delta = 2R^2 \sin A \sin B \sin C$. Hence

$$\frac{\Delta_1}{\Delta} = \frac{R^2 - OP^2}{4R^2} = \frac{\rho}{2R}$$

5. *To show that* $\frac{qr}{p'} = \frac{rp}{q'} = \frac{pq}{r'} = \rho.$

That the first three quantities are equal follows from the equations (A). Let each of them be equal to k. Then, by ratios

$$k = \frac{qr \sin A + rp \sin B + pq \sin C}{p' \sin A + q' \sin B + r' \sin C}$$
$$= \frac{2\Delta_1}{p' \sin A + q' \sin B + r' \sin C} = \frac{4R\Delta_1}{ap' + bq' + cr'}$$
$$= \frac{2R\Delta_1}{\Delta} = \rho,$$

by the corollary at the end of Art. 4.

Hence the proposition is proved.

Similarly,

$$\frac{q'r'}{p} = \frac{r'p'}{q} = \frac{p'q'}{r} = \rho'.$$

6. To show that $\frac{pl}{l'} = \frac{qm}{m'} = \frac{rn}{n'} = \rho.$

Multiplying the first equation in (B) by p, we get $\frac{pl}{l'} = \frac{pq}{r'}$,

Hence $\frac{pl}{l'} = \rho$, and likewise $\frac{qm}{m'}$ and $\frac{rn}{n'}$ are each equal to ρ.

Similarly, we have

$$\frac{p'l'}{l} = \frac{q'm'}{m} = \frac{r'n'}{n} = \rho'.$$

7. *To show that* $pqr = \beta^2\rho$.

We have $\frac{qr}{p'} = \rho$. Multiply by pp' which is equal to β^2. We get $pqr = \beta^2\rho$.

Likewise $p'q'r'=\beta^2\rho'$

8. *To show that* $\beta^2=\rho\rho'$.

Multiplying $pqr=\beta^2\rho$ and $p'q'r'=\beta^2\rho'$, and using equations (A), we get $\beta^6=\beta^4\,\rho\rho'$; or $\beta^2=\rho\rho'$.

Cor. 1. Using the values of ρ and ρ' from the definition in Art. 1, we get

$$(R^2-OP^2)\,(R^2-OP'^2)=4\,R^2\beta^2$$

which is the condition that triangles inscribed in a circle of centre O and radius R, may circumscribe a conic of foci P,P′ and semi minor axis β.

Cor. 2. Making P coincide with the in- and ex-centres, we get the relations

$$R^2-OI^2=2Rr,\ OI_1^2-R^2=2Rr_1,\ \&c.$$

Cor. 3. More generally, if the foci P,P′ of an *inscribed* ellipse be equidistant from O, we get $R^2-OP^2=2R\beta$; and, if the ellipse is an *escribed* one, the relation is $OP^2-R^2=2R\beta$.

Cor. 4. $\qquad pqr=\rho^2\rho',\ p'q'r'=\rho'^2\rho.$

Cor. 5. $\qquad \dfrac{\rho}{\beta}=\sqrt{\dfrac{lmn}{l'm'n'}}$ and $\dfrac{\rho'}{\beta}=\sqrt{\dfrac{l'm'n'}{lmn}}$

For, $\rho:\beta::\beta:\rho'$, and therefore

$$\frac{\rho}{\beta}=\sqrt{\frac{\rho}{\rho'}}=\sqrt{\frac{pqr}{p'q'r'}}=\sqrt{\frac{lmn}{l'm'n'}}.$$

Similarly, $\qquad \dfrac{\rho'}{\beta}=\sqrt{\dfrac{l'm'n'}{lmn}}$

9. *To show that* $4R\alpha\beta=\sqrt{lmn\,l'm'n'}$

We have $lmn=4R\alpha\rho$, and $l'm'n'=4R\alpha\rho'$ (Art 3, Cor.) Multiplying and using the relation $\beta^2=\rho\rho'$, we get the result stated.

Cor. 1. The area of the inscribed ellipse of foci P,P′ is $\dfrac{\pi}{4R}\sqrt{lmn\,l'm'n'}$.

Cor. 2. If I be the incentre, then $IA.\,IB.\,IC=4Rr^2$.

10. *To show that* $\gamma=4R\sin PAB.\sin PBC.\sin PCA$.

We have
$$\beta^2\rho=pqr$$
$$4R\alpha\rho=lmn$$
$$\therefore\quad \frac{\beta^2}{\alpha}=4R.\,\frac{pqr}{lmn}$$

that is,
$$\gamma=4R\sin PAB.\sin PBC.\sin PCA,$$

which is a simple formula for the semi-latus rectum of the inscribed conic of focus P.

Cor. 1. The radius of the inscribed circle of the triangle ABC is

$$4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}.$$

Cor. 2. The radii of the escribed circles are

$$4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}, \&c.$$

Cor. 3. The semi-latus rectum of the inscribed conic whose focus is O, is

$$4R \cos A \cos B \cos C.$$

Cor. 4. The semi-latus rectum of the Brocard ellipse $=4R \sin^2 \omega$.

Cor. 5. The semi-latus rectum of the inscribed parabola whose focus is any point P on the circumcircle of ABC is $4R\cos\lambda\cos\mu\cos\nu$, where λ, μ, ν are the angles which the Simson line of P makes with the sides. [Use the property that the isogonals of PA, PB, PC are in this case paralled to one another and perpendicular to the Simson line of P. This relation, I believe, has been given by Mr. W. Gallatly in this Journal.]

11. It is curious to observe that if p, q, r and ρ be given, then p', q', r', ρ' can be found. For the product pqr divided by ρ gives β^2 and β^2 divided by p, q, r, ρ respectively gives p' q', r', ρ'.

12. If Δ_2 be the pedal triangle of Δ_1 with respect to P (that is to say, the second pedal triangle of ABC with respect to P), to show that the circumradius of $\Delta_2 = \frac{1}{2}\rho$.

We can get this by using the formula $lmn = 2\ \alpha.\lambda$ (Art, 3). Note that Δ_2 stands in the same relation to Δ_1 that Δ_1 stands to Δ. Now, to lmn in the triangle Δ, corresponds pqr in the triangle Δ_1. Also, to λ in the triangle Δ, corresponds β^2 in the triangle Δ_1 as is readily seen geometrically from the common pedal circle of P, P′.

Therefore if k be the circumradius of Δ_2, we have $k = \dfrac{pqr}{2\ \beta^2} = \frac{1}{2}\ \rho$.

Cor. Hence, given a circle and a point P, if any triangle whatever be inscribed in the circle its second pedal triangle with respect to P, has a fixed circumradius.

13. If Δ_3 be the pedal triangle of Δ_2 with respect to P (that is to say, the third pedal triangle of ABC with respect to P), then, the circumradius of $\Delta_3 = \frac{1}{4}\ \gamma$.

We have shown that $lmn = 4\ R\ \alpha\rho$.

$\therefore\ lmn = 2R.2\alpha.2(\frac{1}{2}\rho)$; that is,

$=$product of diameters of circumcircles of Δ, Δ_1, Δ_2.

Applying the result to the triangle Δ_1, if k be the circumradius of Δ_3, we get

$$pqr = 2\alpha . 2\left(\tfrac{1}{2}\rho\right) . 2k$$
$$= 4\alpha\rho k$$
$$\therefore k = \frac{pqr}{4\alpha\rho} = \frac{\beta^2}{4\alpha} = \tfrac{1}{4}\gamma.$$

Cor. If any triangle whatever be circumscribed to a given conic, focus S, then its third pedal triangle with respect to S has a fixed circumradius.

14. It may be observed that the third pedal triangle Δ_3 is similar to the original triangle ABC, P being the centre of similitude and the ratio of similitude being $\mu = ppr/lmn$.* This can be proved easily. From this property of similitude it follows, if Δ_n be the nth pedal triangle of the triangle ABC with respect to P, then, Δ_4 is similar to Δ_1 Δ_5 to Δ_2, and so on, the ratio of similitude being $\mu = (pqr)/(lmn) = \gamma/4R$. Also, the first negative pedal triangle Δ_{-1} is similar to Δ_2, the second negative pedal triangle Δ_{-2} to Δ_1, and so on, the ratio of similitude being $1/\mu = 4R/\gamma$. Hence the circumradius of any desired pedal triangle Δ_n can be found.

15. We shall give another proof of the theorem that $pqr = \beta^2\rho$. For this, we shall first show that

$$ll' \sin A + mm' \sin B + nn' \sin C = 2\Delta.$$

Let $A_1B_1C_1$ be the pedal-triangle of P. Its sides are $l \sin A$, $m \sin B$, $n \sin C$. These are respectively perpendicular to P′A, P′B, P′C. Hence $ll' \sin A$, $mm' \sin B$, $nn' \sin C$ are respectively equal to twice the areas $P'B_1AC_1$, $P'C_1BA_1$, $P'A_1CB_1$ which together make up the triangle ABC. Hence, $ll' \sin A + mm' \sin B + nn' \sin C = 2\Delta$.†

From the foregoing relation we deduce, using equations (A),

$$\frac{ll'}{rr'} \sin A + \frac{mm'}{pp'} \sin B + \frac{nn'}{qq'} \sin C = \frac{2\Delta}{\beta^2}$$

But, by equations (B)

$$\frac{l'}{r'} = \frac{l}{q}, \quad \frac{m'}{p'} = \frac{m}{r}, \quad \frac{n'}{q'} = \frac{n}{q}.$$

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* More generally, given an n sided polygon Σ, and a point P, the nth pedal polygon of Σ with respect to P is similar to Σ, P being the centre of similitude and the ratio of similitude being $\Pi(p)/\Pi(l)$, where the p's are the perpendiculars from P on the sides, and the l's the distances of P from the vertices, of Σ.

† The result can obviously be extended to a polygon circumscribing a conic whose foci are P and P′.

Hence $$\frac{l^2}{qr}\sin A+\frac{m^2}{rp}\sin B+\frac{n^2}{pq}\sin C=\frac{2\triangle}{\beta^2}$$

Multiplying by 2 Rpqr,

$$l^2.ap+m^2.bq+n^2.\,cr=\frac{4\triangle R\,pqr}{\beta^2}.$$

But ap, bq, cr are the multiples for which P is the mean centre of the vertices. Hence, by a well known theorem in Mean Centre,

$$\text{the left hand side}=(ap+bq+cr)\,(R^2-OP^2)$$
$$=2\triangle(R^2-OP^2)$$

$$\therefore\quad \frac{4\triangle R\,.\,pqr}{\beta^2}=2\triangle\,(R^2-OP^2)$$

$$\therefore\quad pqr=\beta^2\,.\frac{R^2-OP^2}{2R}=\beta^2\rho.$$

Contact Circle touching the Nine-point Circle

By M. Bhimasena Rao.

Let S be the centre of a conic touching the sides of a triangle ABC at D,E,F. We shall call the circle through D,E,F, the contact circle of the point S with respect to the triangle ABC.

Let the contact circle cut the sides of ABC again in D′,E′,F′. Let Q be the point of concurrence of AD, BE, CF and Q′ that of AD′, BE′, CF′. Consider the rectangular hyperbola through A,B,C,Q. As DEF is the diagonal triangle of the quadrangle ABCQ, the rect. hyp. ABCQ is self conjugate with respect to DEF and therefore its centre lies on the circle DEF. Similarly, the centre of the rect, hyp. ABCQ′ lies on the circle DEF. Hence the points of intersection of the nine point circle of ABC and the circle DEF are the centres of the rectangular hyperbolas ABCQ and ABCQ′. When the two hyperbolas coincide it is evident that the circle DEF touches the nine-point circle.

2 To find the locus of the point S when its contact circle touches the nine-point circle.

Let P be the isotomic conjugate of Q with respect to ABC. We know that P and S are corresponding points of ABC and its medial triangle. When Q moves on the rect. hyp. ABCQQ′, P describes the line which is its isotomic transformation and S a line passing through the symmedian point of ABC. Hence if S′ be the centre of the inconic D′E′F′, the symmedian point is collinear with S and S′. Also S and S′ are isogonal conjugate points with respect to ABC. (see : *Lemma* on p. 80, J. I. M. S., Vol. III). Therefore the contact circles of isogonal conjugate points on lines through the symmedian point touch the nine-point circle.

The locus of S is the cubic

$$\begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \frac{1}{\alpha} & \frac{1}{\beta} & \frac{1}{\gamma} \end{vmatrix} = o$$

i.e., $a\alpha(\beta^2-\gamma^2)+b\beta(\gamma^2-\alpha^2)+c\gamma(\alpha^2-\beta^2)=0.$

3. Let T be the point of contact of the nine-point circle and the contact circle. The reciprocal polar of the contact circle with respect to the rect. hyp. ABCQQ′ is a parabola inscribed in the triangles DEF and D′E′F′. Its focus is T, the centre of the rect. hyp. The triangles DEF and D′E′F′ being in perspective with ABC, any inconic of

ABC is self conjugate with respect to ABC. Therefore the directrix of the parabola is the isogonal transformation of the rect. hyp. ABCQQ′. (*Vide*, Cor., p. 30 : J. I. M. S., Vol. II).

By question 192 of Mr. V. Ramaswami Aiyar, the two triangles DEF and D′E′F′ are related as in question 178 of Mr. S. Narayanan. The common point of intersection of the six lines, *viz.*—the pedal lines of D,E,F with respect to D′E′F′ and the pedal lines of D′,E′,F′ with respect to DEF, lies on the directrix of the parabola and is equidistant from the ortho-centres of DEF and D′E′F′. (This is proved in Mr. Narayanan's note on 'Concurrent Pedal lines' in the Journal for Dec. 1910 and in Mr. W. Gallatly's note therein referred to. For an extension of this property see my Question 291 in the last number)

When S is the symmedian point, Q becomes the ortho-centre, Q′ the median point; the rect. hyp. is Kiepert's hyperbola, the directrix of the parabola is the Brocard line, and T is the Steiner point of the medial triangle; and it is well known that the pedal lines of the feet of the perpendiculars with respect to the medial triangle and the pedal lines of the middle points of the sides with respect to the ortho-centric triangle intersect at the centre of Taylor's circle.

From the above follow the theorems :

(1) The circumcentre of a triangle, its symmedian point and the ortho-centre of its ortho-centric triangle are collinear (Tucker).

(2) The circumcentre of a triangle and the ortho-centre of its ortho-centric triangle are equidistant from and collinear with the centre of Taylor's circle (Neuberg).

4. If from the centre of the contact circle tangents are drawn to the rect. hyp. ABCQQ′, the pedal circle of either point of contact touches the nine point circle.

Proof : the polar of the centre of the contact circle with respect to the conic of reciprocation ABCQQ′ is the polar of the centre of reciprocation T, with respect to the parabola which is the polar reciprocal. T being the focus of the parabola, the polar of T (the directrix of the parabola) passes through the points of contact of tangents to ABCQQ′ from the centre of the contact circle. Since the directrix is the isogonal conjugate of ABCQQ′ the points of contact of the tangents are isogonal conjugate points with respect to ABC. By M'Cay's extension of Feuerbach's theorem it follows that the pedal circle of isogonal conjugate points on a rect. hyp. circumscribing the triangle touches the nine point circle at the centre of the rect. hyp. Hence the above result.

5. To find the locus of the centre of the contact circle touching the nine point circle.

Let the rect. hyp. ABCQQ′ be

$$p/\alpha+q/\beta+r/\gamma=0$$

then $$p\cos A+q\cos B+r\cos C=0 \qquad (1)$$

The centre of the contact circle has been shown above to be the pole with respect to the hyperbola of its isogonal transformation, *viz*,

$$p\alpha+q\beta+r\gamma=0$$

and is therefore given by

$$\frac{q\gamma+r\beta}{p}=\frac{r\alpha+p\gamma}{q}=\frac{p\beta+q\alpha}{r} \qquad (2)$$

These equations are sufficient to determine the locus.

Since $p\alpha+q\beta+r\gamma=0$ passes through (cos A, cos B, cos C), the centre of the contact circle lies on the polar of (cos A, cos B, cos C) *viz.*,

$$p(\gamma\cos B+\beta\cos C)+\dots\dots=0 \qquad (3)$$

(1) and (3) give the ratios $p:q:r$. Substituting in one of the equations represented by (2) we get the required locus, which may be written

$$\sum(\alpha\cos A).\sum\alpha^2(\cos^2B-\cos^2C)$$
$$+2\sum\alpha\cos A(\beta^2-\gamma^2)(\cos^2A+\cos^2B-\cos^2C)=0,$$

which shows that the curve passes through the *in* and *ex*-centres.

SHORT NOTES.

In-conics.

Let an ellipse inscribed in a triangle be projected into a circle and let points in the projection corresponding to points in the plane ABC be denoted by suffixes .]

The angle of projection $\theta = \cos^{-1} q/p$: $2p$, $2r$ being the axes of the ellipse, and q the inradius of $A_1B_1C_1$.

Then if ω and ω_1 be the centre and its projection,

$\triangle B_1 \omega_1 C_1 = \triangle B \omega C . \cos \theta$

$\therefore q\, a_1 = a\, \alpha \times q/p$

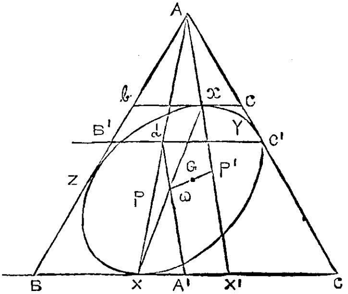
$\therefore a_1 = B_1C_1 = a\alpha/p$

$\therefore 16 \; \triangle_1^2 = (a_1 + b_1 + c_1)\,(\;\;)\,(\;\;)\,(\;\;)$

$$= \frac{(a\,\alpha + b\,\beta + c\,\gamma)\,(\;\;)\,(\;\;)\,(\;\;)}{p^4}.$$

But $\quad \triangle_1 = \triangle \cos \theta = \triangle . q/p.$

And, if $\alpha^1 \beta^1 \gamma^1$ be the $A^1B^1C^1$-normal c's of ω_1



then $\quad -a\,\alpha + b\,\beta + c\gamma = 2\,a\,\alpha^1$

$\therefore abc.\ \alpha^1 \beta^1 \gamma^1 = p^2 q^2 . \triangle$

Hence, if the area of the ellipse has the constant value πk^2, the centre describes the cubic

$$abc\ .\ \alpha^1 \beta^1 \gamma^1 = k^4 \triangle,$$

with $A^1B^1C^1$ as the triangle of reference.

2. The ellipse touching the sides in X, Y, Z, let AX meet B^1C^1 in d $X\omega$ meet the ellipse in x, and A X meet BC in X^1.

Since $X\omega = \omega X$, and $Xd = dA$, it follows that $d\omega$ is parallel to AxX^1. and $d\omega = \frac{1}{2} . Ax$:

And it may easily be shown, by reference to the projection, that ω lies on A^1d:

Hence $$A^1X = A^1X^1.$$

3. Again, $BX : CX = B_1X_1 : C_1X_1$ (in projection)

$$= s_1 - b_1 : s_1 - c_1,$$
$$= a\alpha - b\beta + c\gamma : a\alpha + b\beta - c\gamma,$$
$$= b\beta^1 : c\gamma^1.$$

Therefore AX, BY, CZ are concurrent at a point P whose ABC barycentric c's are $1/a\alpha^1, 1/b\beta^1 : 1/c\gamma^1$.

Again, $BX^1 : CX^1 = CX : BX = c\gamma^1 : b\beta^1$

Hence, AX^1, BY^1, CZ^1 meet at a point P^1 whose bar. c's are $a\,\alpha^1, b\beta^1$ $c\gamma^1$: so that ω and P^1 are homologous points in the $\triangle$'s $A^1B^1C^1$, ABC.

It follows that ωP^1 passes through G: and $GP^1 = 2 \,.\, G\omega$.

The barycentric equation of the ellipse is

$$\sqrt{xx^1} + \sqrt{yy^1} + \sqrt{zz^1} = 0.$$

4. Since ABC is projected orthogonally into $A_1B_1C_1$; and θ is the angle between the planes ABC and $A_1B_1C_1$, we have

$$a_1^2 \cot A + b_1^2 \cot B + c_1^2 \cot C = 2\triangle_1 (\sec\theta + \cos\theta) = 2\triangle(1 + \cos^2\theta)$$

(*vide*: Modern Geometry: p. 77)

But from § 1,

$$pa_1 = a\alpha : \text{ and } \cos\theta = q/p.$$

$$\therefore \frac{abc}{4\triangle^2}(a \cos A \,.\, \alpha^2 + b \cos B \,.\, \beta^2 + c \cos C \,.\, \gamma^2) = p^2 + q^2.$$

The left side is the power of $(\alpha\beta\gamma)$, the centre of the ellipse, for the Polar Circle $\Sigma^2 (a \cos A.\ \alpha) = 0$; [Whitworth: p. 468].

Hence, the Director Circle of an in-conic cuts the Polar Circle orthogonally: a new proof of a well-known theorem.

W. GALLATLY.

The Equilateral Triangle by Paper-folding.

1. The construction given by Mr. Sundara Row for an equilateral triangle by paper-folding is as follows: (page 10, §25, of the American Edition, 1901): "Take a line AB. Bisect it perpendicularly by a line CD. On the bisector take a point D so that AD = AB. Then ADB is the equilateral triangle." This construction practically means the use of the compass though without actually naming it; for, the finding of the point D in the above construction is nothing but a trial method, otherwise.

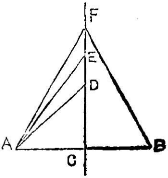
2. The essence of the method of paper-folding consists in the following constructions: (1) joining two points, and bisecting the distance between them perpendicularly, (2) bisecting a given angle and in particular drawing a perpendicular to a given line at a given point, (3) transferring distances from one *given* line to another *given* line.

By a *rigorous* use of this method therefore, it does not seem possible to construct a triangle whose sides are *geometrically* given; of course, if the sides are given integers, then it can be constructed by a previous arithmetical calculation.

3. It is, however, possible to construct an *equilateral* triangle on a given base AB, by paper folding only, thus:—

Bisect the base perpendicularly by CD; take CD = AC. Next, take CE = AD and again, take CF = AE. Then AFB is the required equilateral triangle.

All the steps of this construction can be carried out by using only the three constructions mentioned above.



4. By continuing the above process it will be seen that we can easily construct $a\sqrt{n}$ where a is a given line and n is an integer.

5. Several others of the constructions given by Mr. Sundara Row are open to similar criticisms. In particular, the construction of the pentagon virtually carried out in § 60, and that of a right-angled triangle with a given altitude in § 62. Similarly, it is apparently impossible to find a mean proportional between two given straight lines unless their lengths are given integers, except by the method of trial, which means a virtual use of the compass.

Of course, the above is a criticism of the *rigorousness* of the constructions. It does not at all detract from the use of method by trial from an educational point of view.

R. P. PARANJPYE.

On Bicircular Quartics.

1. Let U, V, W denote three circles, whose centres are at the vertices of the triangle of reference. Consider the equation

$$\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0. \qquad (1)$$

When rationalized this becomes

$$l^2U^2 + m^2V^2 + n^2W^2 - 2mnVW - 2nlWU - 2lmUV = 0;$$

that is, replacing U by $r^2 + u_1 + u_0$, etc.,

$$l^2(r^2 + u_1 + u_0)^2 + \dots\dots - 2mn(r^2 + v_1 + v_0)(r^2 + w_1 + w_0) - \dots = 0$$

Thus, equation (1) denotes a *bicircular quartic*, which degenerates into a *circalar cubic*, when $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$.

Also, the quartic has double contact with the circles U, V, W, as may be readily inferred.

2. Now, the envelope of the system of circles

$$\lambda U+\mu V+\nu W=0 \qquad (2)$$

for varying values of $\lambda\mu\nu$ will coincide with (1), provided

$$l/\lambda+m/\mu+n/\nu=0.$$

But, the centre of (2) is easily seen to be $(\lambda/a, \mu/b, \nu/c)$

Hence, if $(\alpha\ \beta\ \gamma)$ be the centre of (2), the condition that (1) should be the envelope of (2) is

$$l/a\alpha+m/b\beta+n/c\gamma=0 \qquad (3)$$

In other words, the locus of the centres of (2) is the circum-conic (3).

Hence, (3) is the *focal conic* of the quartic.

3. If we interpret equation (1) geometrically, we are lead to the theorem: *If tangents be drawn to the circles U, V, W from any point of the quartic, these tangents are connected by the linear relation,*

$$l^{\frac{1}{2}}t_1+m^{\frac{1}{2}}t_2+n^{\frac{1}{2}}t_3=0. \quad (\textit{Cf. Basset } \S\ 221)$$

M. T. NARANIENGAR.

On the face of the sky for August and September.

(*N.B.*—Standard time is always referred to).

Sideral time at 8 p.m.

		August. H.	M.	S.		*September.* H.	M.	S.
1	...	16	36	18	...	18	38	31
8	...	17	3	54	...	19	6	7
15	...	17	31	30	...	19	33	43
22	...	17	59	7	...	20	1	19
29	...	18	26	42	...	20	28	55

From this table the constellations visible during the evenings of August and September can be ascertained by a reference to their positions as given in a star chart.

Phases of the Moon.

	August.				September.			
	D.	H.	M.		D.	H.	M.	
First Quarter ...	2	5	8	A.M. ...	...			
Full Moon ...	10	8	34	„ ...	8	9	36	P.M.
Last Quarter ...	17	6	0	P.M. ...	15	11	30	„
New Moon ...	24	9	53	A.M. ...	22	8	16	„
First Quarter ...	31	10	0	P.M. ...	30	4	47	„

The Sun.

enters the Autumnal Equinox on Sept. 24 at 9·57 A.M.

The Planets.

Mercury begins as an evening star attaining greatest elongation (27° 25′ E) on Aug. 13 and inferior conjunction on Sept. 19. It is then a morning star attaining greatest elongation (17° 52′ W) on Sept. 25. It is in conjunction with the moon on Aug. 25 and Sept. 21.

Venus continues an evening star attaining greatest brilliancy on Aug. 10 at 6·39 P.M. It is stationary on Aug. 23 and reaches inferior conjunction on Sep. 15 It is in conjunction with the moon on Aug. 26 and Sep. 21.

Mars is in conj· with the Moon on Aug. 17 and Sept. 14 and with Saturn on Aug. 17.

Jupiter is high in the heavens in the evenings. It is in conj. with the Moon on Aug. 2 and 29 and Sep. 26.

Saturn rises about midnight. It is stationary on Sep. 3 when it begins to retrograde. It is in conjunction with the Moon on Aug. 17.

Uranus is in conjunction with the Moon on Aug. 8 and on Sep. 4 at 10.16 P.M.

Neptune is in conjuction with the Moon on Aug. 21 and on Sep. 18.

V. RAMESAM.

SOLUTIONS.

Question 202.

(M. BHIMASENA RAO):—If any three points be so situated that their pedal circles with respect to a triangle have a common radical axis, show that their contact circles with respect to the triangle have also a common radical axis.

Solution by S. Purushotham.

Let D,E,F be three points, such that their pedal circles are coaxal. Then D,E,F and their isogonal conjugates lie three by three on four straight lines. This follows from Q. 253 (1).

Let D′,E′,F′ be the isogonal conjugates, and let the contact circles of D and E interest in P and Q.

Then, by question 175 in the Journal for December 1909, the locus of points whose contact circles pass through P, is a cubic which must be such that its isogonal transformation is itself. (*Vide*: J. I. M. S., Vol III, p. 80).

The most general equation of such a cubic is

$$p\alpha(\beta^2+\gamma^2)+q\beta(\alpha^2+\gamma^2)+r\gamma(\alpha^2+\beta^2)+s\alpha\beta\gamma=0.$$

The cubic passes through D, E. Hence, it passes through F′, and therefore, also, through F its isogonal conjugate.

Thus, the contact circle of F passes through P. Similarly, it can be shown that it passes through Q.

Hence, the result.

Question 251.

(SELECTED):—If a=edge of a regular icosahedron, V its volume, r the radius of the in-sphere, and R the radius of the circum-sphere, shew that

$$r=a\sqrt{\frac{7+3\sqrt{5}}{24}}\,;\ R=a\sqrt{\frac{5+\sqrt{5}}{8}}\,;\ V=\frac{5a^3}{12}(3+\sqrt{5}).$$

Solution by N. Ganapathi Subba Aiyar, B.A., L.T.

If m be the number of sides in each face, n the number of plane angles in a solid angle, a an edge, F the number of faces, and I the inclination of two adjacent faces, we have

(1) $$\sin\frac{I}{2}=\cos\frac{180°}{n}\Big/\sin\frac{180°}{m}$$

(2) $$r=\frac{a}{2}\tan\frac{I}{2}\cot\frac{180^\circ}{m}$$

(3) $$R=\frac{a}{2}\tan\frac{I}{2}\tan\frac{180^\circ}{n}$$

(4) $$V=\frac{1}{3}mF\tan\frac{I}{2}\cot^2\frac{180^\circ}{m}\left(\frac{a}{2}\right)^3$$

Now, in an icosahedron, $m=3$, $n=5$, and $F=20$.

$\therefore$ (1) $\sin\frac{I}{2}=\frac{\cos 36^\circ}{\sin 60^\circ}=\frac{\sqrt{5}+1}{2\sqrt{3}}$, and $\therefore$ $\tan\frac{I}{2}=\frac{\sqrt{5}+1}{\sqrt{5}-1}$

(2) $$r=\frac{a}{2}\cdot\frac{\sqrt{5}+1}{\sqrt{5}-1}\cdot\frac{1}{\sqrt{3}}=\frac{a(3+\sqrt{5})}{4\sqrt{3}}=a\sqrt{\frac{7+3\sqrt{5}}{24}}$$

(7) $$R=\frac{u}{2}\cdot\frac{\sqrt{5}+1}{\sqrt{5}-1}\cdot\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}=v\sqrt{\frac{5+\sqrt{5}}{8}}$$

(4) $$V=\frac{1}{3}\cdot 3\cdot 20\cdot\frac{\sqrt{5}+1}{\sqrt{5}-1}\cdot\frac{1}{3}\cdot\frac{a^3}{8}=\frac{5a^3}{6}\cdot\frac{\sqrt{5}+1}{\sqrt{5}-1}$$

$$=\frac{5a^3}{6}\cdot\frac{6+2\sqrt{5}}{4}=\frac{5a^3}{12}\left(3+\sqrt{5}\right)$$

Question 254. (1)

(V. Ramaswami Aiyar M.A.) :—In any triangle show that

$$OK^2+2PK^2=OP^2\,(1-\tan^2\omega),$$

where O,K,P are the circum centre, the symmedian point, and the orthocentre, and ω is the Brocard angle.

Solution (1) by S. Purushottam, (2) by S. Narayanan, B.A., L.T.

(1) Let the orthocentroidal circle, centre Q, cut the altitudes in L, M, and N. Then the triangle LMN is similar to ABC and has the same symmedian point, *i.e.*, the same *m.c.* for multiples a^2, b^2, c^2, at the vertices. (Lachlan, p. 147, Ex. 2.)

Hence, $QK^2=\frac{1}{9}\,OP^2\,(1-3\tan^2\omega)$, where ω is the Brocard angle of LMN or ABC.

$\therefore$ $OK^2+2PK^2=\frac{2}{3}OP^2+3QK^2=OP^2\,(1-\tan^2\omega)$.

(2) The trilinear coordinates of O,K, P being respectively
(R cosA, RcosB, RcosC), (RsinAtanω, RsinBtanω, R sin C tan ω), and (2 R cos B cos C, 2 R cos C cos A, 2 R cos A cos B), we have

$$OK^2=\frac{\sum\sin 2A\,(R\cos A-R\sin A\tan\omega)^2}{2\Pi\sin A}$$

$$PK^2 = \frac{\sum \sin 2A\,(2R\cos B\cos C - R\sin A\tan\omega)^2}{2\Pi\sin A}$$

Write P_1, P_2 for sinAsinBsinC, cosAcosBcosC, and t for $\tan\omega = P_1/(1+P_2)$, then, after some reduction

$$2P_1\,(OK^2+2PK^2)/R^2 = 2P_1(1+4P_2)+6t^2.P_1(1+4P_2) - 8t(P_1^2+3P_2+3P_2^2),$$
$$= 2P_1(1-8P_2)(1-t^2),$$

$\therefore\ OK^2+2\,PK^2 = OP^2(1-\tan^2\omega)$, since $OP^2 = R^2(1-8P_2)$

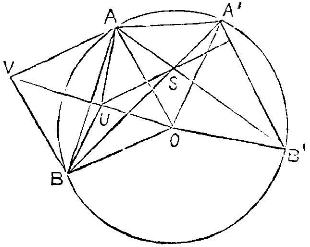
Question 271.

(M. T. NARANIENGAR M.A.) :—AB, A′B′ are two chords of a circle with centre O. A′B, and AB′ intersect in S and the diameter bisecting AB meets the perpendicular to A′B′ through S in U. Show that U is the circumcentre of ASB. Also, if V be the inverse of U with respect to the circle O, prove that the triangles AVB, A′OB′ are similar.

Solution by T. Rajaram Rao, B.A., B.L., and others.

(i) $ASU = \frac{\pi}{2} - \angle AB'A'$

$= \frac{\pi}{2} - ABS$



Hence, U is the circumcentre of ASB.

(ii) Join AU, UB and AO, OB.; and also AA′.

Then $OU.OV = OB^2$ or AO^2,

$\therefore\ UVB = UBO$ or UAO

$\therefore\ AVB = AUB - AOB = 2ASB - 2AA'B$

$= 2A'AB' = A'OB.'$

Hence, $\triangle AVB$ is similar to $\triangle A'OB'$, both being isosceles.

Question 272.

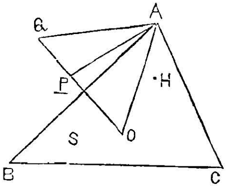
(M. T. Naraniengar M.A.) :—If S, H be the foci of an inconic of a triangle ABC, shew that the circumcentres of ASB, AHB are inverse points with respect to the circle ABC.

Solution by K. J. Sanjana, M.A , and T. Rajarama Rao, B.A., B.L.

Let O,P,Q be the circumcentres respectively of ABC, AHB, ASB on the perpendicular bisector of AB.

Now, as P is the centre of AHB, ∠APQ = ∠AHB. Similarly ∠AQP = π − ∠ASB. But ∠OAP = ∠APQ − ∠AOP = ∠AHB − ∠ACB = ∠HAC + ∠HBC = ∠SAB + ∠SBA = π − ∠ASB

Thus ∠AQP = ∠OAP.



Hence, the triangles OAP, OQA are similar[2] and $OP.\ OQ = OA^2$.

Additional solution by S. Narayan, B.A., L.T.

Question 277.

(*Communicated by* Mr. H. B. Master):—In how many ways can a rupee be exchanged for smaller coins in accordance with the following table? 3 pies = 1 pice, 2 half-pice = 1 pice. 2 pice = 1 half-anna, 2 half-annas = 1 anna, 2 annas = 1 two anna-piece, 2 two-anna pieces = 1 quarter-rupee, 2 quarter-rupees = 1 half-rupee, 2 half-rupees = 1 rupee.

Solution (1) by A. C. L. Wilkinson M.A., and M. K. Sadasiva Aiyar, B.A., B.E. and (2) by Jamshedji Edulji and K. J. Sanjana.

(1) If a denotes a half pie, the coefficient of x^{384a} in the product

$$(1+x^{2a}+x^{4a}+x^{6a}+-)(1+x^{3a}+x^{6a}+\ldots)(1+x^{6a}+\ldots)\ldots(1+x^{192a}+\ldots),$$

where the indices $3a$, $6a$, $12a$, $24a$, $48a$, $96a$, $192a$, form a geometrical progression, is the required number.

We thus require the coefficient of x^{384} in

$$\frac{1}{(1-x^2)(1-x^3)(1-x^6)\ldots(1-x^{192})}$$

or, the coefficient of x^{384} in

$$\frac{1}{(1-x^2)(1-x^3)(1-x^6)(1-x^{12})\ldots\text{to } \infty}$$

diminished by unity.

Now, if $\dfrac{1}{(1-y)(1-y^2)(1-y^4)\ldots\text{to } \infty}=1+a_1y+a_2y^2+\ldots+a_ny^n+\ldots$

then $1+a_1y^2+a_2y^4+\ldots+a_ny^n+\ldots$

$$=(1-y)(1+a_1y+a_2y^2+\ldots+a_ny^n+\ldots);$$

whence $\quad a_{2n}-a_{2n-1}=a_n$

$\quad a_{2n-1}-a_{2n-2}=0$

Thus $\quad a_{2n}=1+a_1+a_2+\ldots+a_n$

$\quad a_{2n-1}=a_{2n-2}$

We deduce

$$a_1=1,\ a_2=2=a_3,\ a_4=4=a_5,\ a_6=6=a_7,\ a_8=10=a_9,$$
$$a_{64}=1828=a_{65},\ldots a_{128}=27338=a_{129}$$

and the number we require is easily seen to be *one* less than

$$1+a_2+a_4+a_6+\ldots+a_{128}=359,671.$$

Hence 359, 670 is the required number.

(2) Let the nos. of the coins in order be $3x_1$, $2x_2, x_3, x_4, x_5, x_6, x_7, x_8$; we have then to find the number of solutions in integral values (including zero) of the equation $x_1+x_3+x_3+2x_4+4x_5+8x_6+16x_7+32x_8=64$.

The no. of solutions, s_2, of $x_1+x_2=64$ is 65. ⊙ The number s_3 of solutions of $x_1+x_2+x_2=64$ may be obtained by giving to x_3 each possible value and solving for the corresponding value of x_1+x_2: this gives $s_3=65+64+63\ldots+1=\frac{1}{2}\cdot 65^2+\frac{1}{2}\cdot 65$. Similarly s_4 may be obtained by giving to x_4 in the equation $x_1+x_2+x_3+2x_4=64$, all possible values *viz.* 0, 1, 2,……32.

This gives $s_3=\frac{1}{2}(65^2+65)\div\frac{1}{2}(63^2+63)+\frac{1}{2}(61^2+61)\ldots+\frac{1}{2}(1^2+1)$
$=\frac{1}{2}\left\{\frac{4}{3}\cdot 33^3-\frac{1}{3}\cdot 33+33^2\right\}=\frac{2}{3}\cdot 33^3+\frac{1}{2}\cdot 33^2-\frac{1}{6}\cdot 33.$

So also, in $r_1+x_2+x_3+2x_4+4x_5=64$, giving to x_5 all possible values *viz.*, 0, 1, 2,…16, we get $s_5=\frac{2}{3}(33^3+31^3\ldots=1^3)+\frac{1}{2}(33^2+31^2\ldots+1^2)-\frac{1}{6}(33+31+\ldots+1)=\frac{2}{3}(2\cdot 17^4-17^2)+\frac{1}{2}(\frac{4}{3}\cdot 17^3-\frac{1}{3}\cdot 17)-\frac{1}{6}\cdot 17^2=\frac{4}{3}\cdot 17,$ $+\frac{2}{3}\cdot 17-\frac{5}{6}\cdot 17^2-\frac{1}{6}\cdot 17.$

Proceding exactly as above, we get successively,

$$s_6=\frac{4}{3}(17^4+15^4\ldots\ldots+1)+\frac{2}{3}(17^3+15^3\ldots\ldots+1)-\frac{5}{6}(17^2+15^2\ldots\ldots+1)$$
$$-\frac{1}{6}(17+15\ldots\ldots+1)=\frac{64}{15}\cdot 9^5+\frac{4}{3}\cdot 9^4-\frac{14}{3}\cdot 9^3-\frac{5}{6}\cdot 9^2+\frac{9}{10}\cdot 9;$$

$$s_7=\frac{64}{15}\left(\frac{16}{3}\cdot 5^6-\frac{20}{3}\cdot 5^4+\frac{7}{3}\cdot 5^2\right)+\frac{4}{3}\left(\frac{16}{5}\cdot 5^5-\frac{8}{3}\cdot 5^3+\frac{7}{15}\cdot 5\right)$$

$$-\frac{14}{3}(2.5^4-5^2)-\frac{5}{6}\left(\frac{4}{3}.5^3-\frac{1}{3}.5\right)+\frac{9}{10}(5^2)=\frac{1024}{45}.5^6+\frac{64}{15}.5^5-\frac{340}{9}.5^4$$
$$-\frac{14}{3}.5^3+\frac{1397}{90}.5^2+\frac{9}{10}.5.$$

From the last we infer that

$$s_8=\frac{1024}{45}(5^6+3^6+1^6)+\frac{64}{15}(5^5+3^5+1^5)-\frac{340}{9}(5^4+3^4+1^4)-\frac{14}{3}$$
$$(5^3+3^3+1^3)+\frac{1397}{90}(5^2+3^2+1^2)+\frac{9}{10}(5+3+1)$$

$=359670$, on actual calculation.

N.B.—The following results can be readily obtained from Bernoulli's formula for the sum of the powers of the natural numbers.

$S_1^n\,(2x-1)=n^2$, $S_1^n\,(2x-1)^2=\frac{4}{3}n^3-\frac{1}{3}n$, $S_1^n\,(2x-1)^3=2n^4-n^2$
$S_1^n\,(2x-1)^4=\frac{16}{5}.n^5-\frac{8}{3}.n^3+\frac{7}{15}.n$, $S_1^n(2x-1)^5=\frac{16}{3}.n^6-\frac{20}{3}.n^4+\frac{7}{3}.n^2$.

Question 279.

(K. J. SANJANA, M.A.) :—A candidate appears at an examination in which there are p papers each carrying m marks. Prove that the greatest number of ways in which any marks could be scored is when the candidate gets 50 per cent. of the total; it being given that no fractional mark is assigned and that mp is an even integer.

Solution by M. K. Sadasiva Aiyar, B.A., B.E.

The solution is the term with the greatest coefficient in the expansion of $(1+x+\ldots+x^m)^p$.

Now, the coefficients in $(1+x+\ldots+x^m)^2$ are obtained by adding to the coefficient of any term in $(1+x+x^2+\ldots x^m)$ those of the preceding m terms, which are each unity.

It is also seen that the coefficients increase gradually up to the central term or terms. Similarly, for the coefficients of $(1+x+\ldots+x^m)^3$, and so on.

Thus we see that when mp is even, the middle term has got the greatest coefficient and the corresponding power of x is $\frac{mp}{2}$. Hence, the question.

Question 281.

(J. C. SWAMINARAYAN, M. A,)—In a triangle ABC, straight lines AX BY, and CZ are drawn, making angles α, β, γ with the sides AB, BC, CA respectively. Shew that the area of the triangles formed by AX, BY and CZ is

$$\frac{2R^2\{\sin(A-\alpha)\sin(B-\beta)\sin(C-\gamma)-\sin\alpha\sin\beta\sin\gamma\}^2}{\sin(B-\beta+\alpha)\sin(C-\gamma+\beta)\sin(A-\alpha+\gamma)}$$

Solution by K. J. Sanjana, M.A,

Let $AZ=a_1$, $AX=a_2$, $BX=b_1$, $BY=b_2$, $CY=c_1$, $CZ=c_2$.

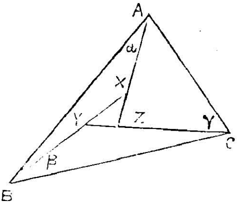
As $\triangle ABC=\Delta_s XYZ+ABX+BCY+CAZ$,

$$\therefore \frac{\triangle ABC}{\triangle XYZ}=1+\frac{\frac{1}{2}XA.XB\sin AXB}{\frac{1}{2}XY.XZ\sin YXZ}+\dots+\dots$$

$$=1+\frac{a_2b_1}{(a_1-a_2)(b_1-b_2)}+\frac{b_2c_1}{(b_1-b_2)(c_1-c_2)}+\frac{c_2a_1}{(c_1-c_2)(a_1-a_2)}$$

But $(a_1-a_2)(b_1-b_2)(c_1-c_2)+a_2b_1(c_1-c_2)+b_2c_1(a_1-a_2)+c_2a_1(b_1-b_2)$ $=a_1b_1c_1-q_2b_2c_2$, identically ; hence

$$\frac{\triangle ABC}{\triangle XYZ}=\frac{a_1b_1c_1-a_2b_2c_2}{XY.\ YZ.\ ZX.}$$



gain

$\sin\beta : \sin(C-\gamma) : \sin Y=c_1 : b_2 : a$,

$\sin\gamma : \sin(A-\alpha) : \sin Z=a_1 : c_2 : b$.

$\sin\alpha : \sin(B-\beta) : \sin X=b_1 : a_2 : c$;

Hence $\sin\alpha\sin\beta\sin\gamma : \sin(A-\alpha)\sin(B-\beta)\sin(C'-\gamma)$ $: \sin X\sin Y\sin Z=a_1b_1c_1 : a_2b_2c_2 : abc$.

Thus we get

$$\frac{\triangle ABC}{\triangle XYZ}=\frac{abc\{\sin\alpha\sin\beta\sin\gamma-\sin(A-\alpha)\sin(B-\beta)\sin(C-\gamma)\}}{XY,\ YZ,\ ZX\sin X\sin Y\sin Z.}$$

But $abc=4R\triangle ABC$, and $XY.\ YZ.\ ZX=\dfrac{8^{\frac{1}{2}}(\triangle XYZ)^{\frac{3}{2}}}{(\sin X\sin Y\sin Z)^{\frac{1}{2}}}$.

Hence $$1=\frac{4R\{\sin\alpha\sin\beta\sin\gamma-\sin(A-\alpha)\sin(B-\beta)\sin(C'-\gamma)\}}{(8\triangle XYZ)^{\frac{1}{2}}(\sin X\sin Y\sin Z)^{\frac{1}{2}}}$$

So that $\triangle XYZ$

$$=\frac{2R^2\{\sin\alpha\sin\beta\sin\gamma-\sin(A-\alpha)\sin(B-\beta)\sin(C'-\gamma)\}^2}{\sin(B-\beta+\alpha)\sin(C'-\gamma+\beta)\sin(A-\alpha+\gamma)}$$

Question 282.

(A. KRISHNASAWMI AIYANGAR) :—From the vertex of the Folium of Descartes, any line is drawn to cut it in any two other points; shew that the intercept between these points subtends a right angle at the origin.

Solution by S. Venkataraya Sastri and others.

The equation of the Folium of Descartes is $x^3+y^3=3axy$, and the vertex A is $\frac{3a}{2}, \frac{3a}{2}$. Therefore $y-\frac{3a}{2}=m\left(x-\frac{3a}{2}\right)$ is the equation of a line through A. If this meets the curve in P,Q, then the homogeneous equation $3(1-m)(x^3+y^3)=2xy(y-mx)$ will be the equation of the three lines joining the origin O to A,P,Q. On simplification this reduces to

$$(x-y)\{x^2(1-m)+xy(1+m)+y^2(m-1)\}=0.$$

But $x-y=0$ is the equation of OA; therefore the equation $x^2(1-m)+xy(1+m)+y^2(m-1)=0$, represents the lines OP, OQ, and this equation represents two straight lines at right angles. Hence, PQ subtends a right angle at the origin.

Question 286.

(UNITY) :—If on the perpendiculars at D, E, F, the midpoints of the sides of ABC, points P, Q, R are taken such that

$$DP/BC=EQ/CA=FR/AB,$$

prove that the triangles PQR, ABC are in perspective and that the axis of perspective envelopes a parabola.

Solution by K. J. Sanjana.

Supposing the triangles PBC, QCA, RAB to be described all outwardly or inwardly, they form directly similar isosceles triangles. Hence if each angle at the base is θ, they are known to be in perspective; and the centre of perspective [cosec $(A\pm\theta)$, cosec $(B\pm\theta)$, cosec $(C\pm\theta)$] is known to lie on Kiepert's hyperbola.

The normal coordinates of Q are $\sin(C+\theta)$, $-\sin\theta$, $\sin(A+\theta)$; those of R, $\sin(B+\theta)$, $\sin(A+\theta)$, $-\sin\theta$, taking the triangles outwards. The equation of QR is

$$x\{\sin^2\theta-\sin^2(A+\theta)\}+y\{\sin(A+\theta)\sin(B+\theta)+\sin\theta\sin(C+\theta)\}$$
$$+z\{\sin(A+\theta)\sin(C+\theta)+\sin\theta\sin(B=\theta)\}=o$$

Thus at the intersection of QR and BC $y/q+z/r=o$, where

$p = \sin(B+\theta)\sin(C+\theta)+\sin\theta\sin(A+\theta)=\sin B\sin C+\sin A\ \sin\ \theta$ with similar values for q and r. Similarly, for the points of meeting of the other corresponding sides.

Therefore the axis of perspective is given by

$$x/p+y/q+z/r=0$$

Putting s for $\sin 2\theta$, and s_1, s_2, s_3 for sin A, sin B, sin C respectively, we have

$$s^2(s_2s_3x+s_3s_1y+s_1s_2z)+s\{s_1(s_2^2+s_3^2)x+s_2(s_3^2+s_1^2)y+s_3(s_1^2+s_2^2)z\}$$
$$+s_1s_2s_3(s_1x+s_2y+s_3z)=0$$

The envelope of this $[\Sigma x^2s_1^2(s_2^2+s_3^2)^2+\Sigma 2yzs_2s_3(s_3^2+s_1^2)(s_1^2+s_1^2)$ $+4s_1s_2s_3(s_1x+s_2y+s_3z)(s_2s_3x+s_3s_1y+s_1s_2^2)]$ may be put into the form

$$\lambda^2+\mu^2+\nu^2-2\mu\nu-2\nu\lambda-2\lambda\mu=0,$$

where $\lambda=xs_1(s_2^2-s_3^2)$, $\mu=ys_2(s_3^2-s_2^4)$, $\nu=zs_3)(s_1^2-s_2^2)$. It is readily seen to be a parabola.

Question 287.

(A. C. L. WILKINSON, M.A., F.R.A.S.):—A point moves in a straight line with a velocity which is initially u, and which receives constant increments equal to u at equal intervals T. Prove that the velocity at any time t after the beginning of the motion is

$$(1)\quad \frac{u}{2}+\frac{ut}{T}+\frac{u}{\pi}\sum_1^\infty\frac{1}{m}\sin\frac{2m\pi t}{T},$$

and that the distance traversed is

$$(2)\quad \frac{ut(t+T)}{2T}+\frac{uT}{12}-\frac{uT}{2\pi^2}\sum_1^\infty\frac{1}{m^2}\cos\frac{2m\pi t}{T},$$

(*Whittaker's Modern Analysis*, p. 159, *Ex.* 10.)

Solution (I) by K. J. Sanjana, (II) by tha Proposer.

(I) The example soccurs in Carslaw's book on Fourier's Series Ch. V. The following indirect proof is independent of Fourier's or Lagrange's Formula.

Let the time t be $(n+\theta)$ T, where n is an integer and θ a positive proper fraction : then the velocity at this time is $(n+1)u$. The summation in (1) is $\sum_1^\infty\frac{1}{m}\sin 2m\pi(n+\theta)$, *i,e.*, $\sum_1^\infty\frac{1}{m}\sin(2m\pi\theta)$, or $\sin 2\pi\theta+\frac{1}{2}\sin 4\pi\theta+\frac{1}{3}\sin 6\pi\theta+\ldots\ldots$

Calling this series S_1, and the corresponding series of cosines C_1 we have as usual

$$C_1+\iota S_1=e^{2\pi\theta\iota}+\tfrac{1}{2}e^{4\pi\theta\iota}+\tfrac{1}{3}e^{6\pi\theta\iota}+\ldots\ldots=-\log(1-e^{2\pi\theta\iota})$$
$$=-\log(2\sin^2\pi\theta-2\iota\sin\pi\theta\cos\pi\theta)$$
$$=-\log\{2\sin\pi\theta(\sin\pi\theta-\iota\cos\pi\theta)\}$$
$$=-\log\frac{2\sin\pi\theta}{\sin\pi\theta+\iota\cos\pi\theta}=-\log(2\sin\pi\theta)+\log(\iota e^{-\pi\theta\iota})$$
$$=-\log(2\sin\pi\theta)+\log\iota-\pi\theta\iota=-\log(2\sin\pi\theta)+\iota\left(\frac{\pi}{2}-\pi\theta\right)$$

Thus the sum is $\pi(\tfrac{1}{2}-\theta)$. Substituting this in expression (1), we get $\frac{u}{2}+\frac{ut}{T}+\frac{u}{\pi}\pi(\tfrac{1}{2}-\theta)=u+\frac{ut}{T}-u\left(\frac{t}{T}-n\right)=u(1+n)$, which proves the result.

The summation in (2) is similarly seen to be

$$C_2\equiv\frac{\cos 2\pi\theta}{1^2}+\frac{\cos 4\pi\theta}{2^2}+\frac{\cos 6\pi\theta}{3^2}+\frac{\cos 8\pi\theta}{4^2}+\ldots\ldots$$

Hence $\frac{dC_2}{d\theta}=-2\pi(\sin 2\pi\theta+\tfrac{1}{2}\sin 4\pi\theta\ldots)=\pi^2(2\theta-1)$; and integrating $C_2=K+\pi^2\theta^2-\pi^2\theta$. To find the constant K, put $\theta=O$; then

$$(C_{2o})=\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{3^2}\ldots\ldots=\frac{\pi^2}{6}.$$

Thus finally $C_2=\pi^2(\theta^2-\theta+\tfrac{1}{6})$. Substituting this in expression (2) we get,

$$\frac{ut^2}{2T}+\frac{ut}{2}+\frac{uT}{12}-\frac{uT\theta^2}{2}+\frac{uT\theta}{2}-\frac{uT}{12}$$
$$=\frac{ut^2(n+\theta)}{2t}+\frac{uT}{2}-\frac{u\theta}{2}-(t-nT)+\frac{uT\theta}{2}$$
$$=\frac{ut(n+1)}{2}+\frac{uT\theta(n+1)}{2}$$
$$=\frac{n(n+1)}{2}uT+u(n+1)T\theta,$$

which is known to be the space described.

(II.) I shall discuss first the series $\sum_1^\infty\frac{\sin 2nx}{n}$ (*cp. Ex.* 2). This might also be discussed as in § 78.

Consider an odd function which is $\frac{\pi}{2}-x$ between o and π

$$\therefore\qquad \frac{\pi}{2}-x=\frac{2}{\pi}\sum\sin nx\int_0^\pi\left(\frac{\pi}{2}-x\right)\sin nx\,dx.\ \S 82.$$

$$\text{Coefficient of } \sin nx = \left[-\frac{\pi}{2}\frac{\cos nx}{n} + \frac{x\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{\pi}$$

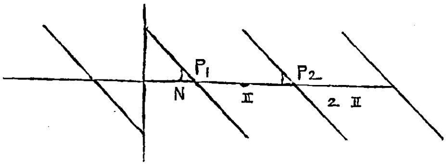
$$= -\frac{\pi}{2}\frac{\cos n\pi - 1}{n} + \frac{\pi\cos n\pi}{n};$$

for n odd, $\cos n\pi = -1$, and the coefficient is zero;

for n even $\cos n\pi = 1$, and the coefficient is $\frac{\pi}{n}$

$$\therefore \quad \frac{\pi}{2} - x = \frac{2}{\pi}\sum_1^{\infty}\frac{\pi}{2m}\sin 2mx, \text{ holds good between o and } \pi$$

Between π and 2π, its value is $\frac{\pi}{2} - x + \pi$. Between $r\pi$ and $\overline{r+1}\,\pi$, its value is $\frac{\pi}{2} - x + r\pi$. Its graph is as follows :



Now $\sum_1^{\infty}\frac{1}{m}\sin 2m x_2 = \sum\frac{1}{m}\sin 2mx_1 = P_1N_1 = \frac{\pi}{2} - x_1$ and $x_2 = x_1 + \pi$

$$= \frac{\pi}{2} - x_2 + \pi \text{ and so on.}$$

For $t = rT + \lambda$, where $\lambda < T, \frac{\pi t}{T} = \pi r + \frac{\pi\lambda}{T}$, lies between $r\pi$ and $(r+1)\pi$.

$$\therefore \frac{u}{2} + \frac{ut}{T} + \frac{u}{\pi}\sum_1^{\infty}\frac{1}{m}\sin\frac{2m\pi t}{T}$$

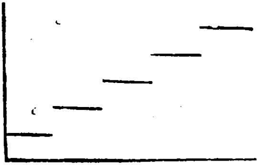
$$= \frac{u}{2} + \frac{ut}{T} + \frac{u}{\pi}\left\{\frac{\pi}{2} - \frac{\pi t}{T} + r\pi\right\} = u + ur = \text{the velocity at the time.}$$

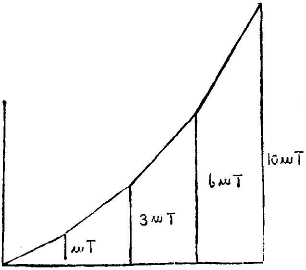
Integrating the series, space described is

$$\frac{ut}{2T}(t+T) - \frac{uT}{2\pi^2}\sum\frac{1}{m^2}\cos\frac{2m\pi t}{T} + C$$

If this vanishes for $t = o$, we have $C = \frac{uT}{12}$.

The velocity is discontinuous and the space continuous. The graphs are as follow :





QUESTIONS FOR SOLUTION.

303. (T. RAJARAMA RAO, B.A., B.L.,) :—Any two conics are drawn with a fixed point S as one of their foci, passing through two other fixed points A and B. P is a fixed point on BA produced. From P tangents PQ and PQ′ are drawn to one of the conics, and tangents PR, PR are drawn to the other conic. Find the locus of the intersection of QR′ and RQ′.

304. (J. C. SWAMINARAYAN, M.A.) :—A particle is repelled from a centre of repulsion O, with a force whose intensity at a distance x is $\frac{\mu}{x^2}-\frac{\lambda}{x^3}$, The particle starts from rest at P, (OP$=a$ and $\mu a>\lambda$). Shew that it will arrive at a point Q in OP produced, (OQ$=b$) in time

$$\frac{a}{2\,\mu a-\lambda}\left[\sqrt{(2\,\mu a-\lambda)\,b^2-2\,\mu a^2 b+a^2\lambda}\right]$$
$$+\frac{\mu a^3}{(2\,\mu a-\lambda)^{\frac{3}{2}}}\cosh^{-1}\frac{(2\,\mu a-\lambda)\,b-\mu a^2}{a\,(\mu a-\lambda)}$$

305. (D. JAGADISAN) :—If the two quadrics $Ax^2+By^2+Cz^2+2\,Fyz+2\,Gzx+2\,Hxy=1$, and $ax^2+by^2+cz^2=1$, touch along a plane section, the plane section will pass through the centre, and will have for its equation $Lx+My+Nz=O$, where $L:M:N$ are given by the equations

$$\frac{L^2+M^2+N^2}{A+B+C-a-b-c}=\frac{(b+c)\,L^2+(c+a)\,M^2+(a+b)\,N^2}{BC+CA+AB-F^2-G^2-H^2-3\,(bc+ca+ab)}$$
$$=\frac{bc\,L^2+ca\,M^2+ab\,N^2}{\Delta-abc.},$$

Δ being the discriminant of the first quadric. Prove also, that the area of the plane section

$$=\pi\sqrt{\frac{A+B+C-a-b-c}{bcA+caB+abC-3abc}}$$

306. (M. T. NARANIENGAR M.A.) :—P, Q are points on a rectangular hyperbola with centre C. If perpendiculars Pp_1, Pp_2 are drawn on CQ and the tangent at Q, prove *geometrically* that $Cp_1=Cp_2$; and genealize the result.

307. (M, T. NARANIENGAR, M.A.) :—Invert a given triangle so that the inverse triangle may be *congruent* to the original.

308. (S. Ramanujan) :—Shew that.

$$(1)\ \int_0^{\frac{\pi}{2}} \theta \cot \theta \log \sin \theta \, d\theta = -\frac{\pi^3}{48} - \frac{\pi}{4}(\log 2)^2.$$

$$(2)\ \int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{x} dx - \tfrac{1}{2}\int_0^1 \frac{\tan^{-1} x}{x} \, dx = \frac{\pi}{8} \log 2.$$

309. (M. Bhimasena Rao.) :—Shew that the Feuerbach's tangents also touch the ellipse which is the locus of the orthopoles of lines through the ortho-centre of the fundamental triangle.

310. (M. K. Sadasiva Aiyar, B.A., B.E.) :—Assuming that the roadway of a bridge is at height h above the centre of the arch-thickness at crown, and that the weight at each point of arch is equal to the weight of the masonry above it up to level of roadway, find the equation of the centre line of arch. How is the equation affected when the dead load due to the weight of road metal is taken into consideration ?

311. (M. K. Sadasiva Aiyar, B.A., B.E.) :—The centering of a brick arch of uniform thickness is struck when the mortar is green. Find the equation of the assumed *intrados*, supposing that the compression due to the wet mortar is proportional to the pressure.

312. (Jamshedji Edalji) :—A Japanese chess-board has 81 cells : they are supposed to be black and white alternately. Determine a route by which a knight starting from a given cell occupies successively *all* the cells *once* and *only once*, and terminates its path in another given cell—both the given cells having the same colour as the four corner cells of the chess-board.

313, (A. Krishnaswami Iyengar) :—TQ, TQ′ are tangents to the ellipse whose centre is C and foci are S,S.′ If the circle TQQ′ touches the ellipse at P, prove that

(1) TP is half the central chord of curvature at P,

(2) the centre of curvature at P lies on the circle TQQ′,

(3) T lies on the circle SPS′,

(4) CP . CT $=$ CS^2,

314. (Selected) :—**41 men are arranged in a ring, which closes up its ranks as every *third* individual is picked out, beginning at a particular position. If a certain named one should be left ultimately, find his position relative to the first one counted.**

315. (G. Ramachandran) :—Construct a triangle having given the incentre, the mid-point of the base, and the foot of the perpendicular from the vertex on the base.

316. (V. Ramaswami Aiyar, M.A.) :—O is the centre and PQ any diameter of the circle inscribed in an equilateral triangle ABC. D is a point in BC such that AO and OD are equally inclined to PQ. Prove that the feet of the perpendiculars from O on the sides of the quadrilateral APDQ lie on a circle which touches the inscribed circle.

317. (R. N. Apte, M.A., F.R.A.S.) :—A particle, starting from rest and moving along a curve, under the action of a force, reaches a certain point in the same time from whatever point on the curve it starts. The rectangular components of the force acting on the particle are μx, μy respectively, Obtaining the equation to the family of these curves, and considering the system of these curves through a given point (x, y) in a given direction y', take the osculating parabolas of each of these curves at the point (x, y), and show that the locus of their foci is two straight lines. (A particular case of Kasner's Theorem).